Published by Institute of Physics Publishing for SISSA
Received: *May 25, 2008*
Accepted: *June 27, 2008*
Published: *July 15, 2008*

Giant magnons in TsT-transformed $\mathrm{AdS}_5 \times \mathrm{S}^5$

Dmitri Bykov[ab] **and Sergey Frolov**[a*]

[a] *School of Mathematics, Trinity College, Dublin 2, Ireland*
[b] *Steklov Mathematical Institute, Moscow, Russia*
E-mail: dbykov@maths.tcd.ie, frolovs@maths.tcd.ie

Abstract: We consider giant magnons propagating in a γ-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background obtained from $\mathrm{AdS}_5 \times \mathrm{S}^5$ by means of a chain of TsT transformations. We point out that in the light-cone gauge and in the infinite J limit the deformed and undeformed string models share the same magnon dispersion relation, the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$-invariant world-sheet S-matrix and the dressing factor. The γ-dependence in the limit is only due to different level-matching conditions. We consider the reduction of the deformed model to $R \times S^3$ and determine the leading γ-dependence of the dispersion relation for a finite J giant magnon.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence.

[*] Correspondent fellow at Steklov Mathematical Institute, Moscow.

JHEP07(2008)071

Contents

1. Introduction and summary

An interesting example of the AdS/CFT duality [1] between gauge and string theory models with reduced supersymmetry is provided by an exactly marginal deformation of $\mathcal{N} = 4$ super Yang-Mills theory [2] and string theory on a deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background suggested in [3]. The deformed models depend on a continuous complex parameter β, and are often called β-deformed. If $\beta \equiv \gamma$ is real the deformed string background can be derived from $\mathrm{AdS}_5 \times \mathrm{S}^5$ by using a TsT transformation which is a combination of a T-duality on one angle variable, a shift of another isometry variable, followed by the second T-duality on the first angle [3, 4]. Moreover, since S^5 has three isometry directions, a chain of TsT transformations can be used to construct a regular three-parameter deformation of $\mathrm{AdS}_5 \times \mathrm{S}^5$ dual to a non-supersymmetric deformation of $\mathcal{N} = 4$ SYM [4]. The Lagrangian of the γ_i-deformed gauge theory can be obtained from the undeformed one by replacing the usual product by the associative $*$-product [3 – 5]. The resulting model is conformal in the planar limit to any order of perturbation theory [6].

Another important property of a TsT transformation is that it preserves the classical integrability of string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [4]. In particular the Lax pair for strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [7] and a TsT transformation can be used to find a Lax pair for strings on a deformed background [4, 8]. Moreover, the Green-Schwarz action for strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is mapped under a TsT transformation to a string action on the γ-deformed background providing a nontrivial example of non-supersymmetric Green-Schwarz action for strings on RR backgrounds [8]. In fact in the Hamiltonian (first-order) formalism the Green-Schwarz action for strings on the γ-deformed background is canonically equivalent to the action for strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ satisfying quasi-periodic or twisted boundary conditions [4, 8]. The twists however are quite unusual because they depend on charges carried by a string and are given by linear combinations of products of the deformation parameters and $\mathfrak{su}(4)$ charges.

This also implies that in the light-cone gauges of [9, 10] the string dynamics on both the γ-deformed background and $\mathrm{AdS}_5 \times \mathrm{S}^5$ is described by the same Hamiltonian density. The γ-dependence enters only through the twisted boundary conditions and the level-matching condition which is modified because a closed string in the deformed background in general corresponds to an open string in $\mathrm{AdS}_5 \times \mathrm{S}^5$. Correspondingly, in the decompactification limit where one of the $\mathfrak{su}(4)$ charges, say J, is sent to infinity while the string tension and the deformation parameters are kept fixed the dependence of the light-cone Hamiltonian on the deformation parameters disappears because in this limit all physical fields must vanish at the space infinity.[1] As a result, if one considers the light-cone gauge-fixed string sigma model off-shell, that is if one does not impose the level-matching condition then the deformed string model is indistinguishable from the undeformed one, and they share the same magnon dispersion relation [18], the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$-invariant world-sheet S-matrix [19 – 21] and the dressing factor [22]–[25]. Therefore, the γ-dependence in the decompactification limit is only due to the level-matching condition.

Thus, to see the dependence of the off-shell spectrum of the model on the deformation parameters one should analyze it for finite values of the $\mathfrak{su}(4)$ charges. The leading dependence can then be captured by the asymptotic Bethe ansatz which would differ from the usual one [26] only by the twists reflecting the non-periodic boundary conditions for finite J. This conclusion is also confirmed by the one-loop considerations in the γ-deformed gauge theory [27, 28, 5] where it is shown that the one-loop integrability of $\mathcal{N} = 4$ SYM [29] is preserved by the deformation, and the corresponding one-loop Bethe ansatz involves the same twists that appear in string theory [5]. In the asymptotic approximation the dispersion relation is not modified and the twists lead to a very mild modification of the string spectrum which basically reduces to γ-dependent shifts of string mode numbers, see [3, 17, 30] for some examples.

The asymptotic Bethe ansatz is not exact and for finite J one expects to find a non-trivial γ-dependence already in the large string tension limit where classical string considerations can be used. In particular, it is interesting to determine how the dispersion relation for a giant magnon [31] depends on the deformation parameters. In the infinite J limit a giant magnon is dual to a gauge theory spin chain magnon, and in the conformal gauge it can be identified with an open string solution of the sigma model reduced to $R \times S^2$. The end-points of the open string move along the equator of S^2 parametrized by an angle ϕ, and the momentum p carried by the dual spin chain magnon is equal to the difference in the angle ϕ between the two end-points of the string [31]. On the other hand in a light-cone gauge a giant magnon is identified with a world-sheet soliton and the momentum p is equal to the world-sheet momentum $p_{\rm ws}$ of the soliton [32]. For finite J the equality between p and $p_{\rm ws}$ holds only in the light-cone gauge $t = \tau\,,\ p_\phi = 1$ [32].

In this paper we determine the leading γ-dependence of the dispersion relation for a finite J giant magnon. We use the conformal gauge and the string sigma model reduced to $R \times S^3$ which in the deformed case is the smallest consistent reduction due to the twisted boundary conditions. Even for the three-parameter deformation the reduced model depends

[1] A γ-dependence remains in the pp-wave [11] and spinning string [12] limits because in these limits the effective length $J/\sqrt{\lambda}$ and the twists $\sim \gamma_i J_k$ are kept fixed, and therefore the string sigma model is defined on a circle with fields obeying quasi-periodic boundary conditions. The pp-wave limits of the deformed backgrounds were discussed in [13 – 15], and the finite-gap integral equations [16] describing spinning strings in the γ-deformed $\mathfrak{su}(2)$ sector were derived in [17].

only on one of the parameters which we denote γ. Since there are two isometry angles ϕ_1 and ϕ_2 a solution of the reduced model can have two non-vanishing charges J_1 and J_2. A giant magnon is then an open string solution of the model which carries only one charge $J \equiv J_1$. The momentum p of the magnon is correspondingly identified with the difference in the angle ϕ_1 between the two end-points of the open string because in the light-cone gauge $t = \tau\,,\ p_{\phi_1} = 1$ it is equal to the world-sheet momentum of a soliton. The second angle ϕ_2 satisfies a twisted boundary condition which can be found by using the general formulas from [4]

$$\Delta\phi_2 = 2\pi(n_2 - \gamma J)\,, \quad n_2 \in \mathbf{Z}\,,$$

where n_2 is an integer winding number of the string in the second isometry direction of the deformed sphere S^3_γ. Collecting all the requirements together, we conclude that a γ-deformed giant magnon can be identified with an open string in $R \times S^3$ satisfying the following conditions

$$\Delta\phi_1 = p\,, \quad \Delta\phi_2 = 2\pi(n_2 - \gamma J)\,, \quad J_1 = J\,, \quad J_2 = 0\,.$$

We analyze the equations of motion and find that a solution exists only for one integer n_2 which obeys the condition $|n_2 - \gamma J| \le \frac{1}{2}$, and therefore there is only one deformation of a giant magnon solution in $R \times S^2$. Then, the leading correction to the dispersion relation in the large J limit has the following form

$$E - J = 2g \sin\frac{p}{2}\left(1 - \frac{4}{e^2}\,\sin^2\frac{p}{2}\,\cos\Phi\, e^{-\frac{\mathcal{J}}{\sin p/2}} + \cdots\right)\,, \quad \Phi = \frac{2\pi(n_2 - \gamma J)}{2^{3/2}\cos^3\frac{p}{4}}\,,$$

where $g = \frac{\sqrt{\lambda}}{2\pi}$ is the string tension, and $\mathcal{J} = J/g$. The formula reduces in the limit $\gamma \to 0$ (or $\Phi \to 0$) to the one obtained in [32]. In the large J limit the γ-dependence disappears in agreement with the discussion above, and if γ is kept fixed then the winding number n_2 goes to infinity too.

The deformed theory has less supersymmetry, and one expects that the energy of a γ-deformed magnon would be higher than the energy of the undeformed one with the same momentum and charge. It is indeed the case because $\cos\Phi < 1$.

It would be interesting to understand how to reproduce the dispersion relation by using Lüscher's approach [33]. This would generalize the computation performed in [34] to the deformed case. The dispersion relation has a peculiar γ-dependence for finite J, and it is not quite clear how such a dependence follows from the S-matrix approach. This would require to generalize Lüscher's formulas to the case of the nontrivial twisted boundary conditions.

Our consideration can be generalized to solutions carrying several spins, see [35 – 37] for recent discussions of the undeformed model. It would be also interesting to compute the one-loop quantum correction generalizing the considerations in [38, 39].

In section 2 we discuss possible giant magnon solutions in the deformed background and explain how they can be mapped to open strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$. In section 3 we sketch the derivation of the leading correction to the dispersion relation in the large J limit and discuss its structure. The details of the derivation can be found in appendix.

JHEP07(2008)071

2. The γ-deformed giant magnon

The bosonic part of the Green-Schwarz action for strings on the γ-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background [8] reduced to $R \times S^5_\gamma$ can be written in the following form

$$S = -\frac{g}{2}\int_{-r}^{r} d\sigma d\tau \Big[\gamma^{\alpha\beta}\left(-\partial_\alpha t \partial_\beta t + \partial_\alpha \rho_i \partial_\beta \rho_i + G\,\rho_i^2 \partial_\alpha \varphi_i \partial_\beta \varphi_i + G\,\rho_1^2\rho_2^2\rho_3^2 \left(\hat{\gamma}_i \partial_\alpha \varphi_i\right)\left(\hat{\gamma}_j \partial_\beta \varphi_j\right)\right)$$
$$- 2G\epsilon^{\alpha\beta}\left(\hat{\gamma}_3 \rho_1^2\rho_2^2 \partial_\alpha \varphi_1 \partial_\beta \varphi_2 + \hat{\gamma}_1 \rho_2^2\rho_3^2 \partial_\alpha \varphi_2 \partial_\beta \varphi_3 + \hat{\gamma}_2 \rho_3^2\rho_1^2 \partial_\alpha \varphi_3 \partial_\beta \varphi_1\right)\Big]. \quad (2.1)$$

Here $g = \frac{R^2}{\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$ is the string tension, and $\gamma^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}$ where $h^{\alpha\beta}$ is a world-sheet metric with Minkowski signature. The function G is defined as follows

$$G^{-1} = 1 + \hat{\gamma}_3^2 \rho_1^2 \rho_2^2 + \hat{\gamma}_1^2 \rho_2^2 \rho_3^2 + \hat{\gamma}_2^2 \rho_1^2 \rho_3^2\,, \qquad \sum_{i=1}^{3} \rho_i^2 = 1\,, \quad (2.2)$$

and φ_i are the three isometry angles of the deformed S^5_γ. The deformation parameters $\hat{\gamma}_i$ are kept fixed in the string sigma model perturbation theory, and are related to the parameters γ_i which appear in the dual gauge theory as $\hat{\gamma}_i = 2\pi g \gamma_i = \sqrt{\lambda}\gamma_i$. The standard $\mathrm{AdS}_5 \times \mathrm{S}^5$ background is recovered after setting the deformation parameters $\hat{\gamma}_i$ to zero. For equal $\hat{\gamma}_i = \hat{\gamma}$ this becomes the supersymmetric background of [3], and the deformation parameter γ enters the $\mathcal{N} = 1$ SYM superpotential as follows $W = h\,\mathrm{tr}(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2)$.

The TsT transformations that map the $AdS_5 \times S^5$ string theory to the γ_i-deformed string theory allow one to relate the angle variables ϕ_i of S^5 to the angle variables φ_i of the γ-deformed geometry. The relations take their simplest form being expressed in terms of the momenta p_i, π_i conjugate to ϕ_i, φ_i, respectively[2] [4]

$$p_i = \pi_i\,, \quad (2.3)$$
$$\rho_i^2\, \phi_i' = \rho_i^2(\varphi_i' - 2\pi\epsilon_{ijk}\gamma_j p_k)\,, \qquad i = 1, 2, 3\,, \quad (2.4)$$

where in (2.4) we sum only in j, k. The relation (2.3) implies that the $\mathrm{U}(1)$ charges $J_i = \int d\sigma p_i$ are invariant under a TsT transformation.

Assuming that none of the "radii" ρ_i vanish on a string solution, we get

$$\phi_i' = \varphi_i' - 2\pi\epsilon_{ijk}\gamma_j p_k\,. \quad (2.5)$$

Integrating eq. (2.5) and taking into account that

$$\Delta\varphi_i = \varphi_i(r) - \varphi_i(-r) = 2\pi n_i\,, n_i \in \mathbb{Z} \quad (2.6)$$

for a closed string in the γ-deformed background, we obtain the twisted boundary conditions for the angle variables ϕ_i of the original S^5 space

$$\Delta\phi_i = \phi_i(r) - \phi_i(-r) = 2\pi(n_i - \nu_i)\,, \quad \nu_i = \epsilon_{ijk}\gamma_j J_k\,, \quad J_i = \int_{-r}^{r} d\sigma\; p_i. \quad (2.7)$$

[2] Here we use definitions of momenta p_i, which differ by a factor of 2π from those of [4], therefore we have an extra 2π in (2.4).

It is clear that if the twists ν_i are not integer then a closed string in the deformed geometry is mapped to an open string in $\mathrm{AdS}_5 \times \mathrm{S}^5$. A giant magnon solution in this respect does not differ essentially from a closed string in $\mathrm{AdS}_5\times\mathrm{S}^5_\gamma$. It corresponds to an open string in the deformed geometry, and its image in $\mathrm{AdS}_5 \times \mathrm{S}^5$ is an open string too. The only difference is that not all of the winding numbers n_i are integer for a giant magnon solution. In fact one linear combination of the winding numbers should be identified with the momentum p carried by the giant magnon.

To determine the linear combination we notice that in the infinite $J \equiv J_1+J_2+J_3$ limit the end-points of a giant magnon should move with the speed of light along a null geodesic of the background [31]. In the undeformed case any geodesics is just a big circle of S^5, and the solution is described by a soliton of the string sigma model reduced to $R \times S^2$. The momentum carried by the soliton is identified with the difference in the angle ϕ between the two end-points of the string where ϕ parametrizes the equator of S^2 [31]. In the light cone gauge $t = \tau\,,\ p_\phi = 1$ the momentum p is equal to the world-sheet momentum of the giant magnon solution and because of that the identification can be also used for finite J [32].

In the γ-deformed background there are infinitely many inequivalent geodesics which correspond to solutions of the Neumann-Rosochatius integrable system [30] (which also describes multi-spin string solutions [40, 41]), and one should choose only those which give the minimum energy satisfying the BPS condition $E = J$. These geodesics were described in [30] where it was shown that for generic values of γ_i there are three BPS states which have only one of the three charges J_i nonvanishing. Choosing for definiteness the nonvanishing charge to be $J_1 = J$, the BPS state corresponds to the geodesics parametrized by the angle φ_1 and having $\rho_1 = 1\,,\ \rho_2 = \rho_3 = 0$. An infinite J giant magnon with the end-points moving along the geodesics is then a solution of the string sigma model reduced to $R\times S^3_\gamma$ where S^3_γ is obtained from the deformed S^5_γ by setting $\rho_3 = 0$. The momentum p carried by the soliton is identified with the difference $\Delta\varphi_1 = \varphi_1(r) - \varphi_1(-r)$. In fact it is easy to see that the TsT transformation maps the infinite J giant magnon solution of the undeformed model to the γ-deformed giant magnon, and therefore the infinite J dispersion relation is not modified, and has no γ dependence. For finite J however the dispersion relation gets a nontrivial γ-dependence which we determine in the next section. This follows from the fact that for the magnon solution $J_2 = J_3 = 0$, and therefore the twist $\nu_1 = 0$, and the corresponding angles ϕ_1 and ϕ_2 of the undeformed S^3 satisfy the following twisted boundary conditions

$$\Delta\phi_1 = \phi_1(r) - \phi_1(-r) = p\,, \quad \Delta\phi_2 = \phi_2(r) - \phi_2(-r) = 2\pi(n_2 - \gamma J)\,, \tag{2.8}$$

where $\gamma \equiv \gamma_3\,,\ J \equiv J_1$. As a result the dispersion relation for the finite J γ-deformed giant magnon depends on p, J and $\delta \equiv 2\pi(n_2 - \gamma J)$. To find the dispersion relation one can either use the conformal gauge [31] or the light-cone gauge [32].

Let us also mention that in the case where the deformation parameters satisfy the relations $\gamma_i = c\,k_i$ where c is any real number and k_i are arbitrary integers, there is another family of BPS states with the following charges [30]

$$J_i = k\,k_i \sim \gamma_i\,, \tag{2.9}$$

where (in quantum theory) k is any integer. In particular, in the supersymmetric case $\gamma_i = \gamma$ the BPS states are the states $(J/3, J/3, J/3)$ with three equal charges. Since $J_i \sim \gamma_i$ for these BPS states the twists ν_i vanish and both the γ-deformed giant magnon and its TsT image satisfy the same twisted boundary conditions which take the simplest form in terms of the following new angle variables and their conjugate momenta

$$\psi_1 = k_1\phi_1 + k_2\phi_2 + k_3\phi_3\,, \qquad \pi_1 = \frac{p_1 + p_2 + p_3}{k_1 + k_2 + k_3}\,, \tag{2.10}$$

$$\psi_2 = k_1\phi_1 - (k_1 + k_3)\phi_2 + k_3\phi_3\,, \qquad \pi_2 = \frac{k_2 p_1 - k_1 p_2}{k_1(k_1 + k_2 + k_3)}\,, \tag{2.11}$$

$$\psi_3 = k_1\phi_1 + k_2\phi_2 - (k_1 + k_2)\phi_3\,, \qquad \pi_3 = \frac{k_3 p_1 - k_1 p_3}{k_1(k_1 + k_2 + k_3)}\,. \tag{2.12}$$

Then, the giant magnon solution with the charges satisfying (2.9) satisfies the following boundary conditions

$$\Delta\psi_1 = p\,, \quad \Delta\psi_2 = 0\,, \quad \Delta\psi_3 = 0\,. \tag{2.13}$$

Since the boundary conditions do not depend on γ_i in the classical theory the dispersion relation for the giant magnon does not depend on the deformation parameters either. A disadvantage of this giant magnon solution is that the corresponding Bethe ansatz is not known.

3. Finite J dispersion relation

To determine the dispersion relation we impose the conformal gauge $\gamma^{\alpha\beta} = \mathrm{diag}(-1, 1)$, set $t = \tau$, and use the following parametrization of S^3

$$x_i^2 = 1\,,\ x_1 + ix_2 = \rho_1 e^{i\phi_1}\,,\ x_3 + ix_4 = \rho_2 e^{i\phi_2}\,,\ \rho_2^2 = 1 - \rho_1^2 = \chi\,. \tag{3.1}$$

Then the sigma model action for strings on $R \times S^3$ takes the following form

$$S = -\frac{g}{2}\int_{-r}^{r} d\sigma d\tau \left(\frac{\partial_\alpha\chi\partial^\alpha\chi}{4\chi(1-\chi)} + (1-\chi)\partial_\alpha\phi_1\partial^\alpha\phi_1 + \chi\partial_\alpha\phi_2\partial^\alpha\phi_2\right)\,.$$

and solutions of the equations of motion should also satisfy the Virasoro constraints

$$\frac{\dot\chi^2 + \chi'^2}{4\chi(1-\chi)} + (1-\chi)\left(\dot\phi_1^2 + \phi_1'^2\right) + \chi\left(\dot\phi_2^2 + \phi_2'^2\right) = 1\,, \tag{3.2}$$

$$\frac{\dot\chi\chi'}{4\chi(1-\chi)} + (1-\chi)\dot\phi_1\phi_1' + \chi\dot\phi_2\phi_2' = 0\,. \tag{3.3}$$

Since $t = \tau$ the range of σ is related to the space-time energy E of a solution as follows

$$2r = \frac{E}{g} \equiv \mathcal{E}\,. \tag{3.4}$$

The two charges $J_1 \equiv J$ and J_2 corresponding to shifts of ϕ_1 and ϕ_2 are

$$J = g\int_{-r}^{r} \mathrm{d}\sigma\,(1-\chi)\,\dot\phi_1\,, \quad J_2 = g\int_{-r}^{r} \mathrm{d}\sigma\,\chi\,\dot\phi_2\,. \tag{3.5}$$

As was discussed in the previous section, the γ-deformed giant magnon solution has only one nonvanishing charge J, and the angles ϕ_1 and ϕ_2 satisfy the following twisted boundary conditions

$$\Delta\phi_1 = \phi_1(r) - \phi_1(-r) = p\,, \quad \Delta\phi_2 = \phi_2(r) - \phi_2(-r) = \delta\,, \tag{3.6}$$

where $\delta = 2\pi(n_2 - \gamma J)$, $\gamma = \gamma_3$ and n_2 is the winding number in the φ_2 direction of the deformed S^5_γ . It is worth mentioning that the dependence on γ and n_2 comes only through their linear combination δ which in fact plays the role of the deformation parameter.

The problem of finding a finite J giant magnon solution is thus basically equivalent to the problem of finding a two-spin giant magnon solution discussed in appendix C of [32], and can be solved by using a similar ansatz

$$\phi_1(\sigma,\tau) = \omega\tau + \frac{p}{2r}(\sigma - v\tau) + \phi(\sigma - v\tau)\,, \tag{3.7}$$

$$\phi_2(\sigma,\tau) = \nu\tau + \frac{\delta}{2r}(\sigma - v\tau) + \alpha(\sigma - v\tau)\,, \tag{3.8}$$

$$\chi(\sigma,\tau) = \chi(\sigma - v\tau)\,, \tag{3.9}$$

where $\chi(\sigma)$, $\phi(\sigma)$ and $\alpha(\sigma)$ satisfy the periodic boundary conditions.

Substituting the ansatz into the equations of motion, integrating the equations for ϕ and α once, and using the Virasoro constraint (3.2) , we get the following three equations

$$\phi' = f_0 + \frac{f_1}{1-\chi}\,, \qquad \alpha' = a_0 + \frac{a_1}{\chi}\,, \tag{3.10}$$

$$\kappa^2\,\chi'^2 = (\chi - \chi_{\rm neg})(\chi - \chi_{\rm min})(\chi_{\rm max} - \chi)\,, \tag{3.11}$$

where the constants in the equations are functions of $\omega, \nu, v, p, \delta$, and $\chi_{\rm neg}, \chi_{\rm min}, \chi_{\rm max}$ are ordered as $\chi_{\rm neg} \le 0 \le \chi_{\rm min} < \chi_{\rm max}$. Moreover, giant magnon solutions exist only if $\chi_{\rm max} \le 1$ and for these solutions $\chi_{\rm min} \le \chi \le \chi_{\rm max}$, see appendix for detail.

If the deformation parameter δ goes to 0 then $\chi_{\rm neg}$, a_0 , a_1 approach 0 too, and we recover the equations of motion for a finite J undeformed giant magnon [32].

For any value of δ we can always choose the initial conditions so that $\chi(\sigma)$ is an even function and $\phi(\sigma)$ and $\alpha(\sigma)$ are odd functions of σ, and since they are also periodic functions, we can always look for a solution satisfying the following boundary conditions

$$\begin{aligned} &\chi(-r) = \chi(r) = \chi_{\rm min}\,, && \chi(0) = \chi_{\rm max}\,, && \chi(-\sigma) = \chi(\sigma)\,, \\ &\phi(-r) = \phi(0) = \alpha(-r) = \alpha(0) = 0\,, && \phi(-\sigma) = -\phi(\sigma)\,, && \alpha(-\sigma) = -\alpha(\sigma)\,. \end{aligned} \tag{3.12}$$

Due to the conditions we can restrict our attention to the half of the string from $-r$ to 0, and since χ is an increasing function on this interval we can also replace integrals over σ by integrals over χ from $\chi_{\rm min}$ to $\chi_{\rm max}$. Then a solution is completely determined by the following five equations which are analyzed in detail in appendix

$$\text{Periodicity of } \phi: \qquad r\, f_0 + f_1 \int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{(1-\chi)|\chi'|} = 0\,,$$

$$\text{Periodicity of } \alpha: \qquad r\, a_0 + a_1 \int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{\chi|\chi'|} = 0\,,$$

$$\text{Charge } \mathcal{J} \equiv \frac{J_1}{g}: \qquad -2r\, v\, f_1 + \frac{\omega}{1-v^2} \int\limits_{\chi_{\min}}^{\chi_{\max}} d\chi\, \frac{1-\chi}{|\chi'|} = \mathcal{J}\,,$$

$$\text{Charge } J_2 = 0: \qquad -2r\, v\, a_1 + \frac{\nu}{1-v^2} \int\limits_{\chi_{\min}}^{\chi_{\max}} d\chi\, \frac{\chi}{|\chi'|} = 0\,,$$

$$\text{Length of string:} \qquad \int_{-r}^{0} d\sigma = r = \int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{|\chi'|}\,,$$

where all constants should be expressed in terms of the charge $\mathcal{J}$, the soliton momentum p and the deformation parameter δ.

The dispersion relation can be found in the large $\mathcal{J}$ limit as an expansion in $e^{-\frac{\mathcal{J}}{\sin(p/2)}}$, and up to the first correction it has the following form $(0 \leq p \leq \pi)$

$$E - J = 2g \sin\frac{p}{2} \left(1 - \frac{4}{e^2} \sin^2\frac{p}{2} \cos\Phi\, e^{-\frac{\mathcal{J}}{\sin p/2}} + \cdots \right)\,, \tag{3.13}$$

where

$$\Phi = \frac{\delta}{2^{3/2}\cos^3\frac{p}{4}} = \frac{2\pi(n_2 - \gamma J)}{2^{3/2}\cos^3\frac{p}{4}}\,. \tag{3.14}$$

The dispersion relation in the γ-deformed model reduces in the limit $\delta \to 0$ (or $\Phi \to 0$) to the one obtained in [32].

Some remarks are in order.

1. We see that in the limit $\mathcal{J} \to \infty$ the dispersion relation is independent of the deformation parameter. This is contrary to papers [42, 43] where it was claimed that the momentum is shifted by the deformation parameter $2\pi\gamma$. As was discussed in the previous section, $2\pi\gamma$ is identified with $\hat{\gamma}/g$, and therefore the shift by γ cannot be seen in classical theory in any case. It would be a one-loop effect, and the discussion in the Introduction indicates that the momentum p is not shifted at one loop at all but one should take into account that in quantum theory magnons carry other charges of order one, and therefore $p = \Delta\phi_1$ is not equal to $p_{\rm ws} = \Delta\varphi_1$. According to (2.7), if we have several (or just one) magnons with the total charges J_2, J_3 then the momenta are related as $p = p_{\rm ws} + 2\pi\gamma_3 J_2 - 2\pi\gamma_2 J_3$. If the state is physical then the total world-sheet momentum $p_{\rm ws}$ should vanish leading to the condition $p = 2\pi\gamma_3 J_2 - 2\pi\gamma_2 J_3$ (up to an integer multiple of 2π). This condition is equivalent to the cyclicity constraint in the twisted Bethe ansatz [5].

JHEP07(2008)071

2. Since $\cos\Phi < 1$ the energy of a γ-deformed magnon is higher than the energy of the undeformed one with the same momentum and charge. That is what one should expect because the deformed theory has less supersymmetry.

3. The derivation of the dispersion relation performed in appendix shows that a giant magnon solution exists if Φ satisfies the restriction

$$-\pi \leq \Phi \leq \pi\,, \tag{3.15}$$

and therefore if we require a solution to exist for all values of p from $-\pi$ to π the parameter δ must also satisfy the same restriction

$$-\pi \leq \delta \leq \pi \quad \Longleftrightarrow \quad |n_2 - \gamma J| \leq \frac{1}{2}\,. \tag{3.16}$$

This means that n_2 is the integer closest to γJ. We see that for any γJ there is only one integer n_2 which satisfies the condition, and therefore there is only one deformation of a giant magnon solution in $R \times S^2$. If the fractional part of γJ is less than $1/2$ then n_2 is equal to the integer part of γJ, and if the fractional part of γJ is greater than $1/2$ then n_2 is equal to the integer part of $\gamma J + 1$.

4. For small enough values of p however the first-order perturbation theory in $e^{-\frac{\mathcal{J}}{\sin(p/2)}}$ allows one to have two or three integers satisfying the restriction (3.15): n_2 satisfying (3.16), and $n_2 \pm 1$. We expect that the latter possibilities will be ruled out at higher orders of the perturbation theory. Anyway, according to (3.13) their energies would be higher than the energy of the main solution.

Acknowledgments

We thank Gleb Arutyunov for discussions. The work of D.B. was supported by the EU-RTN network *Constituents, Fundamental Forces and Symmetries of the Universe* (MRTN-CT-2004-512194), in part by grant of RFBR 08-01-00281-a and in part by grant for the Support of Leading Scientific Schools of Russia NSh-795.2008.1. The work of S.F. was supported in part by the Science Foundation Ireland under Grant No. 07/RFP/PHYF104.

A. The motion on γ-deformed S^3

The metric of $AdS_5 \times S^5$, reduced to the $\mathbb{R} \times S^3$ takes the following form:

$$ds^2 = -dt^2 + \frac{d\chi^2}{4\chi(1-\chi)} + (1-\chi)d\phi_1^2 + \chi d\phi_2^2. \tag{A.1}$$

We will be looking for a solution of the equations of motion in the following form:

$$\phi_1(\sigma,\tau) = \omega\tau + \frac{p}{2r}(\sigma - \upsilon\tau) + \phi(\sigma - \upsilon\tau); \tag{A.2}$$

$$\phi_2(\sigma,\tau) = \nu\tau + \frac{\delta}{2r}(\sigma - \upsilon\tau) + \alpha(\sigma - \upsilon\tau); \tag{A.3}$$

$$\chi(\sigma,\tau) = \chi(\sigma - \upsilon\tau), \tag{A.4}$$

where $\delta = 2\pi(n_2 - \gamma J_1)$ and $\phi(\sigma - v\tau)$, $\alpha(\sigma - v\tau), \chi(\sigma - v\tau)$ satisfy periodic boundary conditions.

Substituting the ansatz into the equations of motion, integrating the equations for ϕ and α once, and using the Virasoro constraints (3.2) , we get the following equations:

$$\begin{aligned}
\phi' &= -\left(\frac{v\omega}{1-v^2} + \frac{p}{2r}\right) - \frac{vA_1}{1-v^2}\frac{1}{1-\chi} && \text{(A.5)}\\
\alpha' &= -\left(\frac{v\nu}{1-v^2} + \frac{\delta}{2r}\right) - \frac{vA_2}{1-v^2}\frac{1}{\chi} && \text{(A.6)}\\
\frac{(1-v^2)^2}{4}\chi'^2 &= \kappa_0 + \kappa_1\chi + \kappa_2\chi^2 + \kappa_3\chi^3 && \text{(A.7)}\\
\omega A_1 + \nu A_2 + 1 &= 0. && \text{(A.8)}
\end{aligned}$$

The constants κ_i are as follows:

$$\begin{aligned}
\kappa_0 &= -v^2 A_2^2 && \text{(A.9)}\\
\kappa_1 &= 1 - \omega^2 + v^2(1 + A_2^2 - A_1^2) && \text{(A.10)}\\
\kappa_2 &= -1 - \nu^2 + 2\omega^2 - v^2 && \text{(A.11)}\\
\kappa_3 &= \nu^2 - \omega^2, && \text{(A.12)}
\end{aligned}$$

Thus, in the notation of section 3 one may write

$$\begin{aligned}
f_0 &= -\left(\frac{v\omega}{1-v^2} + \frac{p}{2r}\right); \qquad f_1 = -\frac{vA_1}{1-v^2};\\
a_0 &= -\left(\frac{v\nu}{1-v^2} + \frac{\delta}{2r}\right); \qquad a_1 = -\frac{vA_2}{1-v^2};\\
\kappa &= \frac{1-v^2}{2\sqrt{\omega^2-\nu^2}}.
\end{aligned}$$

We also have the following expressions for the charges:[3]

$$\mathcal{J} = \frac{1}{1-v^2}\left(2rv^2A_1 + \omega\int_{-r}^{r} d\sigma\,(1-\chi)\right) \tag{A.13}$$

$$\mathcal{J}_2 = \frac{1}{1-v^2}\left(2rv^2A_2 + \nu\int_{-r}^{r} d\sigma\,\chi\right) = 0. \tag{A.14}$$

[3]From these expressions one can derive a linear relation between $E, \mathcal{J}, \mathcal{J}_2$:

$$\frac{1-v^2}{\mathcal{E}}\left(\frac{\mathcal{J}}{\omega} + \frac{\mathcal{J}_2}{\nu}\right) = 1 + v^2\left(\frac{A_1}{\omega} + \frac{A_2}{\nu}\right).$$

Our equations can be written in the following form:

$$\text{Periodicity of } \phi: \qquad \frac{rv\omega}{1-v^2}+\frac{p}{2}=-\frac{v\,A_1}{1-v^2}\int\limits_{\chi_{\min}}^{\chi_{\max}}\frac{d\chi}{(1-\chi)|\chi'|}; \tag{A.15}$$

$$\text{Periodicity of } \alpha: \qquad \frac{rv\nu}{1-v^2}+\pi\delta=-\frac{v\,A_2}{1-v^2}\int\limits_{\chi_{\min}}^{\chi_{\max}}\frac{d\chi}{\chi|\chi'|}; \tag{A.16}$$

$$\text{Charge } \mathcal{J}\equiv\frac{J_1}{g}: \qquad \mathcal{J}=\frac{2}{1-v^2}\left(rA_1v^2+\omega\int\limits_{\chi_{\min}}^{\chi_{\max}}d\chi\,\frac{(1-\chi)}{|\chi'|})\right); \tag{A.17}$$

$$\text{Charge } \mathcal{J}_2\equiv\frac{J_2}{g}=0: \qquad 0=rv^2A_2+\nu\int\limits_{\chi_{\min}}^{\chi_{\max}}d\chi\,\frac{\chi}{|\chi'|}, \tag{A.18}$$

and the periodicity condition for χ which in this case takes the form

$$\text{Length of string:} \quad \int_{-r}^{0}d\sigma=r=\int\limits_{\chi_{\min}}^{\chi_{\max}}\frac{d\chi}{|\chi'|}\,. \tag{A.19}$$

We have called the real roots of the equation $\chi_{\rm neg},\chi_{\min},\chi_{\max}$ with the following ordering $\chi_{\rm neg}\le 0\le\chi_{\min}<\chi_{\max}$. Moreover, for the consistency of our approach we have to require that $\chi_{\min},\chi_{\max}\in[0,1)$, which will be justified by the solution. The fact that in the large J expansion one of the roots is negative can be easily proven. Indeed, in the strict $J\to\infty$ limit it follows from the work [32] that $\omega=1,\ \nu=0$, therefore the leading coefficient κ_3 of the polynomial in the r.h.s. of (A.7) is negative, and this should remain true for large J. The value of the r.h.s. of (A.7) at $\chi=0$ is $\kappa_0\le 0$. These two facts together imply that there's a negative root $\chi_{\rm neg}$. Note also that the value of the r.h.s. of (A.7) at $\chi=1$ is $-v^2A_1^2<0$. This, together with the previous observation, implies that the two other roots of the polynomial either are both <0 or both $\in[0,1)$ or both >1. We're interested in the case when they both lie in $[0,1)$. We consider $(\chi_{\rm neg},\chi_{\min},\chi_{\max})$ as independent variables that, together with all the previous variables (ν,ω,v,A_2), satisfy the following conditions which simply mean that $(\chi_{\rm neg},\chi_{\min},\chi_{\max})$ are actually solutions of the cubic equation:

$$\chi_{\rm neg}+\chi_{\min}+\chi_{\max} = -\frac{\kappa_2}{\kappa_3} \tag{A.20}$$

$$\chi_{\rm neg}\chi_{\min}+\chi_{\min}\chi_{\max}+\chi_{\rm neg}\chi_{\max} = \frac{\kappa_1}{\kappa_3} \tag{A.21}$$

$$\chi_{\rm neg}\chi_{\min}\chi_{\max} = -\frac{\kappa_0}{\kappa_3}. \tag{A.22}$$

We now switch to more convenient variables $(\widetilde{v},\epsilon)$ instead of $\chi_{\min},\chi_{\max}$ (leaving $\chi_{\rm neg}$ unaltered). These two sets are connected in the following way:[4]

$$\begin{aligned}
\epsilon &= \frac{\chi_{\min} - \chi_{\rm neg}}{\chi_{\max} - \chi_{\rm neg}}\,; && \text{(A.23)}\\
\widetilde{v}^2 &= \frac{1-\chi_{\max}}{1-\chi_{\rm neg}}\,; && \text{(A.24)}\\
\chi_{\rm neg} &= \chi_{\rm neg}\,. && \text{(A.25)}
\end{aligned}$$

Next we write the expressions for all integrals entering our equations:

$$\begin{aligned}
\int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{\chi|\chi'|} &= \frac{2\kappa}{(1-\widetilde{v}^2)^{3/2}(1-\chi_{\rm neg})^{1/2}(1+\chi_{\rm neg}\frac{\widetilde{v}^2}{1-\widetilde{v}^2})}\Pi\left(\frac{1-\chi_{\rm neg}}{1+\chi_{\rm neg}\frac{\widetilde{v}^2}{1-\widetilde{v}^2}}(1-\epsilon);1-\epsilon\right);\\
\int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{(1-\chi)|\chi'|} &= -\frac{2\kappa}{\widetilde{v}^2(1-\chi_{\rm neg})^{3/2}\sqrt{1-\widetilde{v}^2}}\,\Pi\left(\frac{\widetilde{v}^2-1}{\widetilde{v}^2}(1-\epsilon);1-\epsilon\right); \qquad \text{(A.26)}\\
\int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi}{|\chi'|} &= \frac{2\kappa\, K(1-\epsilon)}{\sqrt{(1-\chi_{\rm neg})(1-\widetilde{v}^2)}};\\
\int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi\,\chi}{|\chi'|} &= 2\kappa\frac{\chi_{\rm neg}\,K(1-\epsilon)+(1-\chi_{\rm neg})(1-\widetilde{v}^2)\,E(1-\epsilon)}{\sqrt{(1-\chi_{\rm neg})(1-\widetilde{v}^2)}};\\
\int\limits_{\chi_{\min}}^{\chi_{\max}} \frac{d\chi\,(1-\chi)}{|\chi'|} &= -2\kappa\frac{(\chi_{\rm neg}-1)\,K(1-\epsilon)+(1-\chi_{\rm neg})(1-\widetilde{v}^2)\,E(1-\epsilon)}{\sqrt{(1-\chi_{\rm neg})(1-\widetilde{v}^2)}}.
\end{aligned}$$

Thus, we have chosen the parameter ϵ rather than J as our expansion parameter. This means that we have to make an expansion of the system of equations (A.15)–(A.19) in ϵ and determine the corresponding coefficients in the expansion of various parameters, comparing powers of ϵ and/or $\log\epsilon$ which arise in this expansion. First of all, before solving the equations, we get rid of the variable r by plugging the expression for r from (A.19) into all other equations.

We make the following ansatz for our parameters:

$$\begin{aligned}
v(\epsilon) &= v_0(\epsilon) + v_1(\epsilon)\epsilon + O(\epsilon^2);\\
\widetilde{v}(\epsilon) &= \widetilde{v}_0(\epsilon) + \widetilde{v}_1(\epsilon)\epsilon + O(\epsilon^2);\\
\omega(\epsilon) &= \omega_0(\epsilon) + \omega_1(\epsilon)\epsilon + O(\epsilon^2);\\
\nu(\epsilon) &= \nu_1(\epsilon)\epsilon + O(\epsilon^2); \qquad \text{(A.27)}\\
A_1(\epsilon) &= A_{1,0}(\epsilon) + A_{1,1}(\epsilon)\epsilon + O(\epsilon^2);\\
A_2(\epsilon) &= A_{2,1}(\epsilon)\epsilon + O(\epsilon^2);
\end{aligned}$$

[4] The purpose of introducing the variable ϵ should be clear — then the moduli of all tori in our expressions become $1-\epsilon$. The purpose of introducing $\widetilde{v}$ is the following: the first parameter of the Π-function in (A.26) becomes $\frac{\widetilde{v}^2-1}{\widetilde{v}^2}(1-\epsilon)$, so that it is in direct correspondence with an analogous parameter in the work [32].

$$\chi_{\rm neg}(\epsilon) = \chi_1(\epsilon)\epsilon + O(\epsilon^2);$$
$$\mathcal{J}(\epsilon) = \mathcal{J}_0(\epsilon) + \mathcal{J}_1(\epsilon)\epsilon + O(\epsilon^2),$$

where we assume that all "coefficient" functions like $v_0(\epsilon), v_1(\epsilon), \widetilde{v}_0(\epsilon)$, etc. are terminating series in $\log\epsilon$ (this is the reason why expansions (A.27) are justified). This assumption will be proved aposteriori — by the solution that we will find.

We substitute (A.27) into our equations and expand these equations in ϵ^m, ignoring terms with logarithms (that is, treating any combination $\left(\sum\limits_{k=0}^{n} a_k(\log\epsilon)^k\right)\epsilon^m$ as just ϵ^m). Then we obtain a system of equations for our "coefficient" functions, which, when solved, exhibits the property of these functions mentioned above — that is, they're terminating series in powers of $\log\epsilon$.

In the course of expanding the above written equations we need an expansion for $\Pi(1-\alpha\,\epsilon, 1-\epsilon)$ as $\epsilon \to 0$ (α fixed and $0 < \alpha < 1$). To find such an expansion we make use of the following textbook identity for elliptic functions:

$$\Pi(1-\alpha\,\epsilon, 1-\epsilon) = \frac{1}{\alpha(\alpha-1)\epsilon}\left[\alpha(1-\epsilon)K(1-\epsilon) - (1-\alpha\epsilon)\Pi\left(\frac{\alpha-1}{\alpha};1-\epsilon\right)\right]. \quad \text{(A.28)}$$

The meaning of using this identity is that it explicitly singles out the $\frac{1}{\epsilon}$ factor in the expansion. Once we have written $\Pi(1-\alpha\,\epsilon, 1-\epsilon)$ in this form, we may use `Mathematica` to generate the expansions of functions in the r.h.s. of (A.28):

$$\begin{aligned}\Pi(1-\alpha\epsilon, 1-\epsilon) = {} & \frac{\arctan\left(\sqrt{\frac{1}{\alpha}-1}\right)}{\sqrt{\frac{1}{\alpha}-1}\,\alpha\,\epsilon} + \qquad \text{(A.29)}\\ & + \frac{\left(2\alpha\sqrt{\frac{1}{\alpha}-1}\arctan\left(\sqrt{\frac{1}{\alpha}-1}\right) + (\alpha-1)(-\log(\epsilon/16)+1)\right)}{4(\alpha-1)} + \\ & + \frac{\left(8\alpha^2\sqrt{\frac{1}{\alpha}-1}\arctan\left(\sqrt{\frac{1}{\alpha}-1}\right) - (\alpha-1)(2\alpha+2(2\alpha+1)\log(\epsilon/16)+3)\right)\epsilon}{64(\alpha-1)} + O\left(\epsilon^2\right)\end{aligned}$$

However, in our case α is not constant in ϵ but rather depends on ϵ in the following way:

$$\alpha(\epsilon) = \frac{\frac{\chi_{\rm neg}(\epsilon)}{\epsilon} + (1-\chi_{\rm neg}(\epsilon))(1-\widetilde{v}^2(\epsilon))}{1-\widetilde{v}^2(\epsilon)(1-\chi_{\rm neg}(\epsilon))}. \quad \text{(A.30)}$$

According to our ansatz (A.27) $\alpha(\epsilon)$ has a finite positive limit smaller than 1 as $\epsilon \to 0$ — this is the only thing, which is important for our expansions to be justified. That is, we plug the expansion of α in (powers and logarithms of) ϵ into the expansion for $\Pi(1-\alpha\,\epsilon, 1-\epsilon)$ obtained at fixed α.

We also need to know the expansion of $\Pi\left(\frac{\widetilde{v}^2-1}{\widetilde{v}^2}(1-\epsilon);1-\epsilon\right)$ as $\epsilon \to 0$. It was constructed in the appendix of [32]. One has to use the identity

$$\Pi\left(\frac{v^2-1}{v^2}(1-\epsilon);1-\epsilon\right) = \tag{A.31}$$
$$= \frac{1}{(1-(1-v^2)\,\epsilon)\,K(\epsilon)}\left[\frac{1}{2}\pi v\sqrt{(1-v^2)\,(1-(1-v^2)\,\epsilon)}F\left(\arcsin\left(\sqrt{1-v^2}\right);\epsilon\right)+\right.$$
$$\left.+K(1-\epsilon)\left(\left(1-\left(1-v^2\right)\epsilon\right)K(\epsilon)-\left(1-v^2\right)(1-\epsilon)\Pi\left(\frac{v^2\epsilon}{1-(1-v^2)\,\epsilon};\epsilon\right)\right)\right]$$

In the r.h.s. there's only one function, which has an expansion that cannot be directly obtained by `Mathematica`, and its expansion looks as follows:

$$\Pi\left(\frac{v^2\epsilon}{1-(1-v^2)\,\epsilon};\epsilon\right) = \frac{\pi}{2}+\frac{1}{8}\left(2\pi v^2+\pi\right)\epsilon+\frac{1}{128}\pi\left(-8v^4+44v^2+9\right)\epsilon^2+$$
$$+\frac{1}{512}\pi\left(16v^6-72v^4+206v^2+25\right)\epsilon^3+O\left(\epsilon^4\right). \tag{A.32}$$

Inverting the expansion

$$J(\epsilon)=J_0(\epsilon)+J_1(\epsilon)\epsilon+o(\epsilon), \tag{A.33}$$

we obtain ϵ as a function of J, that is we return to our original expansion in the limit $J\to\infty$:

$$\epsilon(J)=\frac{16}{e^2}e^{-\frac{\mathcal{J}}{\sin\frac{p}{2}}}\left[1-\frac{8}{e^2}e^{-\frac{\mathcal{J}}{\sin\frac{p}{2}}}\left(1-\mathcal{J}\,\frac{2-3\sin^2\frac{p}{2}}{2\sin\frac{p}{2}}\cos(\Phi)-\frac{1}{2}\mathcal{J}^2\cot^2\frac{p}{2}\cos\Phi\right)+\cdots\right]. \tag{A.34}$$

We now write out explicitly the expansions of the parameters entering the equations of motion:

$$\begin{aligned}
\chi_{\rm neg}(\mathcal{J}) &= -\frac{16}{e^2}\sin^2\frac{p}{2}\,\sin^2\frac{\Phi}{2}\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\cdots, \qquad\qquad (A.35)\\
\chi_{\rm max}(\mathcal{J}) &= \sin^2\frac{p}{2}+\frac{8}{e^2}\sin\frac{p}{2}\,\cos^2\frac{p}{2}\,\cos\Phi\,\left(3\,\sin\frac{p}{2}+\mathcal{J}\right)\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\cdots,\\
\chi_{\rm min}(\mathcal{J}) &= \frac{16}{e^2}\sin^2\frac{p}{2}\,\cos^2\frac{\Phi}{2}\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\cdots,\\
v(\mathcal{J}) &= \cos\frac{p}{2}-\frac{4}{e^2}\sin\frac{p}{2}\,\cos\frac{p}{2}\,\cos\Phi\,\left(\sin\frac{p}{2}+\mathcal{J}\right)\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\cdots,\\
\omega(\mathcal{J}) &= 1+\frac{8}{e^2}\sin^2\frac{p}{2}\,\cos\Phi\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\cdots,\\
\nu(\mathcal{J}) &= \frac{4}{e^2}\cos\frac{p}{2}\,\sin\Phi\,\left(2\sin\frac{p}{2}+\mathcal{J}\right)\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\cdots,\\
f_0(\mathcal{J}) &= -\frac{p}{\mathcal{E}}-\frac{\cos\frac{p}{2}}{\sin^2\frac{p}{2}}+\frac{\cos\Phi\,\sin p\,\left(2\mathcal{J}\cos p+6\mathcal{J}-\sin\frac{p}{2}+3\sin\frac{3p}{2}\right)}{2e^2\,\sin^4\frac{p}{2}}\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\cdots,\\
f_1(\mathcal{J}) &= \frac{\cos\frac{p}{2}}{\sin^2\frac{p}{2}}+\frac{\cos\Phi\,\sin p\,\left(\sin\frac{3p}{2}-2\mathcal{J}(\cos p+3)-11\sin\frac{p}{2}\right)}{2e^2\,\sin^4\frac{p}{2}}\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\cdots,\\
a_0(\mathcal{J}) &= -\frac{\delta}{\mathcal{E}}-\frac{4}{e^2}\left(\mathcal{J}+2\sin\frac{p}{2}\right)\sin\Phi\,\cot^2\frac{p}{2}\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\cdots,\\
a_1(\mathcal{J}) &= \frac{8}{e^2}\sin\frac{p}{2}\,\sin\Phi\,e^{-\frac{\mathcal{J}}{\sin(p/2)}}+\cdots,
\end{aligned}$$

where

$$\Phi = \frac{\delta}{2^{3/2}\cos^3(\frac{p}{4})}, \tag{A.36}$$

and the solution exists for all $p \in [-\pi;\pi]$ (if and) only if

$$|\delta| = |2\pi(n_2 - \gamma J)| \leq \pi. \tag{A.37}$$

This means that for the undeformed $AdS_5 \times S^5$, that is $\gamma = 0$, the only possible choice is $n_2 = 0$, or $\delta = 0$. In this case all formulas reduce to what was found in [32].

To obtain the dispersion relation one should expand (A.19) with respect to ϵ and then substitute the expansion (A.34) of ϵ in terms of J. The dispersion relation with the first correction has the following form:

$$\begin{aligned} E - J &= \frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2}\left(1 - \frac{4}{e^2}\sin^2\frac{p}{2}\cos\Phi\, e^{-\frac{\mathcal{J}}{\sin p/2}} + \cdots\right); \\ \Phi &= \frac{\delta}{2^{3/2}\cos^3\frac{p}{4}}; \quad |\delta| = |2\pi(n_2 - \gamma J)| \leq \pi. \end{aligned} \tag{A.38}$$

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[2] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional $N = 1$ supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 95 [hep-th/9503121].

[3] O. Lunin and J.M. Maldacena, *Deforming field theories with* U(1) × U(1) *global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [hep-th/0502086].

[4] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069 [hep-th/0503201].

[5] N. Beisert and R. Roiban, *Beauty and the twist: the Bethe ansatz for twisted $N = 4$ SYM*, *JHEP* **08** (2005) 039 [hep-th/0505187].

[6] S. Ananth, S. Kovacs and H. Shimada, *Proof of all-order finiteness for planar beta-deformed Yang-Mills*, *JHEP* **01** (2007) 046 [hep-th/0609149]; *Proof of ultra-violet finiteness for a planar non-supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 783** (2007) 227 [hep-th/0702020].

[7] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [hep-th/0305116].

[8] L.F. Alday, G. Arutyunov and S. Frolov, *Green-Schwarz strings in TsT-transformed backgrounds*, *JHEP* **06** (2006) 018 [hep-th/0512253].

[9] G. Arutyunov and S. Frolov, *Integrable Hamiltonian for classical strings on $AdS_5 \times S^5$*, *JHEP* **02** (2005) 059 [hep-th/0411089].

[10] S. Frolov, J. Plefka and M. Zamaklar, *The $AdS_5 \times S^5$ superstring in light-cone gauge and its Bethe equations*, *J. Phys.* **A 39** (2006) 13037 [hep-th/0603008].

[11] D.E. Berenstein, J.M. Maldacena and H.S. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *JHEP* **04** (2002) 013 [hep-th/0202021].

JHEP07(2008)071

[12] S. Frolov and A.A. Tseytlin, *Multi-spin string solutions in $AdS_5 \times S^5$*, *Nucl. Phys.* **B 668** (2003) 77 [hep-th/0304255].

[13] V. Niarchos and N. Prezas, *BMN operators for $N = 1$ superconformal Yang-Mills theories and associated string backgrounds*, *JHEP* **06** (2003) 015 [hep-th/0212111].

[14] R. de Mello Koch, J. Murugan, J. Smolic and M. Smolic, *Deformed PP-waves from the Lunin-Maldacena background*, *JHEP* **08** (2005) 072 [hep-th/0505227].

[15] T. Mateos, *Marginal deformation of $N = 4$ SYM and Penrose limits with continuum spectrum*, *JHEP* **08** (2005) 026 [hep-th/0505243].

[16] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical / quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].

[17] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for superconformal deformations of $N = 4$ super Yang-Mills theory*, *JHEP* **07** (2005) 045 [hep-th/0503192].

[18] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar $N = 4$ super Yang-Mills*, *JHEP* **07** (2004) 075 [hep-th/0405001].

[19] M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054 [hep-th/0412188].

[20] N. Beisert, *The* SU(2|2) *dynamic S-matrix*, hep-th/0511082.

[21] G. Arutyunov, S. Frolov and M. Zamaklar, *The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring*, *JHEP* **04** (2007) 002 [hep-th/0612229].

[22] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [hep-th/0406256].

[23] R. Hernandez and E. Lopez, *Quantum corrections to the string Bethe ansatz*, *JHEP* **07** (2006) 004 [hep-th/0603204].

[24] N. Beisert, R. Hernandez and E. Lopez, *A crossing-symmetric phase for $AdS_5 \times S^5$ strings*, *JHEP* **11** (2006) 070 [hep-th/0609044].

[25] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* (2007) P01021 [hep-th/0610251].

[26] N. Beisert and M. Staudacher, *Long-range* PSU(2, 2|4) *Bethe ansaetze for gauge theory and strings*, *Nucl. Phys.* **B 727** (2005) 1 [hep-th/0504190].

[27] R. Roiban, *On spin chains and field theories*, *JHEP* **09** (2004) 023 [hep-th/0312218].

[28] D. Berenstein and S.A. Cherkis, *Deformations of $N = 4$ SYM and integrable spin chain models*, *Nucl. Phys.* **B 702** (2004) 49 [hep-th/0405215].

[29] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208].

[30] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for (non)supersymmetric deformations of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 731** (2005) 1 [hep-th/0507021].

[31] D.M. Hofman and J.M. Maldacena, *Giant magnons*, *J. Phys.* **A 39** (2006) 13095 [hep-th/0604135].

[32] G. Arutyunov, S. Frolov and M. Zamaklar, *Finite-size effects from giant magnons*, *Nucl. Phys.* **B 778** (2007) 1 [hep-th/0606126].

[33] M. Lüscher, *Volume dependence of the energy spectrum in massive quantum field theories*. 1. *Stable particle states*, *Commun. Math. Phys.* **104** (1986) 177.

[34] R.A. Janik and T. Lukowski, *Wrapping interactions at strong coupling — the giant magnon*, *Phys. Rev.* **D 76** (2007) 126008 [arXiv:0708.2208].

[35] Y. Hatsuda and R. Suzuki, *Finite-size effects for dyonic giant magnons*, *Nucl. Phys.* **B 800** (2008) 349 [arXiv:0801.0747].

[36] J.A. Minahan and O. Ohlsson Sax, *Finite size effects for giant magnons on physical strings*, *Nucl. Phys.* **B 801** (2008) 97 [arXiv:0801.2064].

[37] T. Klose and T. McLoughlin, *Interacting finite-size magnons*, *J. Phys.* **A 41** (2008) 285401 [arXiv:0803.2324].

[38] N. Gromov, S. Schäfer-Nameki and P. Vieira, *Quantum wrapped giant magnon*, arXiv:0801.3671.

[39] M.P. Heller, R.A. Janik and T. Lukowski, *A new derivation of Lüscher F-term and fluctuations around the giant magnon*, *JHEP* **06** (2008) 036 [arXiv:0801.4463].

[40] G. Arutyunov, S. Frolov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$ and integrable systems*, *Nucl. Phys.* **B 671** (2003) 3 [hep-th/0307191].

[41] G. Arutyunov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: new integrable system relations*, *Phys. Rev.* **D 69** (2004) 086009 [hep-th/0311004].

[42] C.-S. Chu, G. Georgiou and V.V. Khoze, *Magnons, classical strings and β-deformations*, *JHEP* **11** (2006) 093 [hep-th/0606220].

[43] N.P. Bobev and R.C. Rashkov, *Multispin giant magnons*, *Phys. Rev.* **D 74** (2006) 046011 [hep-th/0607018].